Practices before the class (March 8)

- (**T/F**) A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.

(T/F)
- If $\dim(\mathrm{Nul}(A)) = 0$, then linear system $Ax = b$ has a unique solution for every b.

Practices before the class (March 8)

- (**T/F**) A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot map $\mathbb{R}^2$ onto $\mathbb{R}^3$.
  True. A maps onto if and only if the columns of A spans $\mathbb{R}^3$. This is impossible because A only has two columns.
  In general, given a linear transformation $T : V \to W$ with $\dim V < \dim W$, then T cannot be onto.
- If $\dim(\mathrm{Nul}(A)) = 0$, then linear system $Ax = b$ has a unique solution for every b.
  False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\operatorname{rank} A = 2$ and by the Rank Theorem,
  $\dim(\mathrm{Nul}(A)) = 0$. But $A\mathbf{x} = \mathbf{b}$ has no solution if $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

5.3 Diagonalization

In many cases, the eigenvalue-eigenvector information contained within a matrix A can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix.

The following example shows that the powers of a diagonal matrix D are easy to compute, so as the matrix A.

Example 1. Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

Then use the formula to compute A^4.

ANS: First notice that the powers of D are easy to compute:

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}, \quad D^3 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}, \ \ldots$$

and $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

To find A^k, we compute:

$$A^2 = A \cdot A = PD\underbrace{P^{-1} \cdot P}_{I}DP^{-1} = PD^2P^{-1}$$

$$A^3 = A \cdot A \cdot A = PD\underbrace{P^{-1}P}_{I}D\underbrace{P^{-1}P}_{I}DP^{-1} = PD^3P^{-1}$$

$$\vdots$$

$$A^k = A \cdot A \cdots A = PD^kP^{-1}. \quad \text{So } A^k = PD^kP^{-1}$$

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{Recall } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 3^k \\ -5^k & -2 \cdot 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ -2 \cdot 5^k + 2 \cdot 3^k & -5^k + 2 \cdot 3^k \end{bmatrix}$$

So $A^4 = \begin{bmatrix} 2 \cdot 5^4 - 3^4 & 5^4 - 3^4 \\ -2 \cdot 5^4 + 2 \cdot 3^4 & -5^4 + 2 \cdot 3^4 \end{bmatrix}$

Definition. A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

The following theorem characterizes diagonalizable matrices and tells how to construct a suitable factorization.

Theorem 5 The Diagonalization Theorem
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Example 2. In the following exercise, the matrix A is factored in the form PDP^{-1}. Use the Diagonalization Theorem to find the eigenvalues of A and a basis for each eigenspace.

$$\underbrace{\begin{bmatrix} 5 & -2 & -2 \\ 1 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & -1 & -2 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{P^{-1}}$$

ANS: By the Diagonalization Theorem,

the eigenvalues and the corresponding basis for the eigenspaces are

$$\lambda = 3 : \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\lambda = 4 : \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

Diagonalizing Matrices

There are four steps to implement the description in Theorem 5. We use the following example to show the algorithm.

Example 3. Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution.

Step 1. Find the eigenvalues of A. Solve $|A - \lambda I| = 0$:

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\left[(\lambda+5)(\lambda-1)+9\right] - 3\left[3(\lambda-1)+9\right] + 3\left[-9+3(5+\lambda)\right]$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) \cancel{-3(3\lambda+6)} \cancel{+3(3\lambda+6)}$$

$$= -(\lambda-1)(\lambda+2)^2$$

$$= 0$$

Thus the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2, -2$,

Step 2. Find three linearly independent eigenvectors of A. This is the critical step. If it fails, then Theorem 5 says that A cannot be diagonalized.

$\lambda_1 = 1$, we solve $(A - \lambda I)\vec{x} = \vec{0}$. The augmented matrix is

$$\begin{bmatrix} A - I & \vec{0} \end{bmatrix} = \begin{bmatrix} 0 & \cancel{3}\,1 & \cancel{3}\,1 & 0 \\ -3 & -6 & -3 & 0 \\ \cancel{3}\,1 & \cancel{3}\,1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \cancel{-3}\,1 & \cancel{-3}\,1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Thus an eigenvector corresponding to $\lambda = 1$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Similarly, we solve $(A - \lambda_2 I)\vec{x} = 0$ for $\lambda_2 = -2$. we find

Basis for $\lambda_2 = -2$: $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

We can check that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an linearly independent set. (One way is to compute determinant of $[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$

Step 3. **Construct P from the vectors in step 2.**

$$P = \begin{array}{c} [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3] \\ \text{corresponds} \uparrow \quad \uparrow \quad \uparrow \\ \lambda_1 \quad \lambda_2 \quad \lambda_2 \end{array} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4. **Construct D from the corresponding eigenvalues.**

It is important that the order of eigenvalues in D matches the eigenvectors in P, i.e.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We can double-check that P and D found work. We can verify that $AP = PD$, which is equivalent to $A = PDP^{-1}$ when P is invertible. (However, be sure that P is invertible!) Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Recall Theorem 2 in $\S 5.1$, which states *eigenvectors corresponding to distinct eigenvalues are linearly independent*. Combining this fact with the *The Diagonalization Theorem*, we have:

Theorem 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Matrices Whose Eigenvalues Are Not Distinct

When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows:

Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.
a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.
b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n, and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.
c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example 4. Diagonalize the matrix, if possible. Check Exercise 7 in this notes for an example of A that is not diagonalizable, which is simliar to Handwritten HW#24, question 17.

$A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 1$

- For $\lambda = 2$: $A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$ and solve $(A-2I)\vec{x} = \vec{0}$

$$[A-2I \quad \vec{0}] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is .

$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and a basis for the eigenspace

is $\{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda=1$ $\quad A-I=\begin{bmatrix}-1 & -4 & -6\\ -1 & -1 & -3\\ 1 & 2 & 4\end{bmatrix}$ and solving $(A-I)\vec{x}=\vec{0}$, we

have $\vec{x}=x_3\begin{bmatrix}-2\\-1\\1\end{bmatrix}$.

Thus a basis for the eigenspace is $\vec{v}_3=\begin{bmatrix}-2\\-1\\1\end{bmatrix}$

Construct

$P=[\vec{v}_1\ \vec{v}_2\ \vec{v}_3]=\begin{bmatrix}-2 & -3 & -2\\ 1 & 0 & -1\\ 0 & 1 & 1\end{bmatrix}$

and set

$D=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$

where the eigenvalues in D corresponds to $\vec{v}_1\ \vec{v}_2\ \vec{v}_3$ respectively.

Exercise 5. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is not diagonalizable? Justify your answer.

Exercise 6. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

Exercise 7. Diagonalize the matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$